Automorphism groups and branch coverings of graphs

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The theory of Riemann surfaces was founded in classical works by B. Riemann and A. Hurwitz. We note that originally Riemann surface was defined as a branched covering over the sphere. Over the last decade, a few discrete versions of the theory of Riemann surfaces were created.

1. Bacher, R., de la Harpe, P., and Nagnibeda,T., 1997
2. Urakawa, H., 2000
3. Baker, M., Norine, S., 2009
4. Caporaso, L., 2011

In these theories, the role of Riemann surfaces is played by graphs. As well as branched coverings are replaces by quasi-coverings of graphs.

Introduction

Dictionary

1. Riemann surface $\Longleftrightarrow$ Finite connected multigraph
2. Holomorphic map $\Longleftrightarrow$ Harmonic map
   (branched covering) (quasi-covering)
3. The sphere $\Longleftrightarrow$ Tree
4. Torus (= one "hole" surface) $\Longleftrightarrow$ Flower (= one cycle graph)
5. Genus ($\sharp$ of "holes") $\Longleftrightarrow$ Genus ($\sharp$ of independent loops)
6. Conformal automorphism $\Longleftrightarrow$ Automorphism acting harmonically
   (= acting free on arcs)

Harmonic maps and branched coverings

One of the first definitions of branched covering for graphs was done by T. D. Parsons, T. Pisanski and B. Jackson (1980). The main idea was to find a discrete version of branched covering for graph through dual voltage assignment.

Following Baker-Norine (2007) we prefer to give the following geometric definition.

For any vertex x of a graph G we denote by St_x the star of G at x.

Definition

A morphism $\varphi : G \to G'$ is called to be branched covering (also quasi-covering, harmonic map and so on in the literature) if for all vertices $x \in V(G)$, the quantity $|\varphi^{-1}(e') \cap St_x|$ is independent of the choice of edge $e' \in E(St_{\varphi(x)})$.

Riemann-Hurwitz formula for graphs

Recall the classical Riemann-Hurwitz formula. Given surjective holomorphic map $\varphi : S \to S'$ between Riemann surfaces of g and g', respectively, one has

$$2g - 2 = \deg(\varphi)(2g' - 2) + \sum_{x \in S}(r_\varphi(x) - 1), \tag{1}$$

where $r_\varphi(x)$ denotes the ramification index of φ at x. Let G be a finite group of conformal automorphisms acting on S and $\varphi : S \to S' = S/G$ is the canonical map induced by the group action. Then the above formula can be rewritten in the form

$$2g - 2 = |G|(2g' - 2) + \sum_{x \in S}(|G^x| - 1), \tag{2}$$

where G^x stands for the stabiliser of x in G and $|G^x| = r_\varphi(x)$ is the order of a stabiliser.

Remark that S has only finite number of points with non-trivial stabiliser.

Riemann-Hurwitz formula for graphs

The latter formula has a natural discrete analogue. By a graph we mean a finite connected multigraph without loops. We define genus of graph X as $g = |E(G)| - |V(G)| + 1$, that is as cyclomatic number of G. Let G be a finite group acting on graph X without fixed and invertible edges. Denote by g' genus of the factor graph $X' = X/G$. Then by [Baker-Norine, 2009] we have

$$g - 1 = |G|(g' - 1) + \sum_{x \in V(X)} (|G^x| - 1), \tag{3}$$

where $V(X)$ is the set of vertices of X.
We extend the above mentioned results to group actions with fixed and invertible edges.

Finite group action on graphs

We say that a group G acts on X if G is a subgroup of $\mathrm{Aut}(X)$. Let X be a finite connected graph. We note the genus $g(X) = |E(X)| - |V(X)| + 1$ coincides with the Betti number of X that is the rank of the first homology group $H_1(X, \mathbb{Z})$.

Let G be a finite group acting on the graph X. An edge $\{x, \bar{x}\} \in E(X)$ consisting of two semi-edges x and $\bar{x}$ is said to be invertible (or reversible) by G if there is an element $g \in G$ such that g sends x to $\bar{x}$ and $\bar{x}$ to x.

An edge $\{x, \bar{x}\} \in E(X)$ is said to be fixed by G if there is a non-trivial element $g \in G$ that fixes x and $\bar{x}$. We say that G acts on X *without edge reversing* if X has no edges invertible by G. Also, G acts on X without fixed edges if X has no edges fixed by G.

Our first result is the following theorem for groups acting on a graph without edge reversing.

Theorem 1 (M., 2013)

Let X be a graph of genus g and G is a finite group acting on X without edge reversing. Denote by $g(X/G)$ genus of the factor graph X/G. Then

$$g-1=|G|(g(X/G)-1)+\sum_{v\in V(X)}(|G^v|-1)-\sum_{e\in E(X)}(|G^e|-1),$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x stands for the stabiliser of $x\in V(X)\cup E(X)$ in G and $|G^x|$ is the order of a stabiliser.

Groups acting on a graph with edge reversing

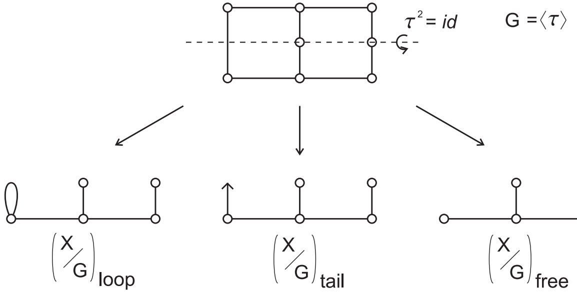
Let now G be a finite group acting on a graph X, possibly with invertible edges. In this case, there are three different ways to define the factor graph X/G.

1°. *The factor graph with loops* $(X/G)_{loop}$.

2°. *The factor graph with semi-edges* $(X/G)_{tail}$

3°. *The factor graph without semi-edges* $(X/G)_{free}$.

Groups acting on a graph with edge reversing



Groups acting on a graph with edge reversing

We have the following two theorems.

Theorem 2 (M., 2013)

Let X be a graph of genus g and G is a finite group acting on X, possibly with edge reversing. Denote by $g(X/G)_{loop}$ genus of the factor graph $(X/G)_{loop}$. Then

$$g-1=|G|(g(X/G)_{loop}-1)+\sum_{v\in V(X)}(|G^{v}|-1)-\sum_{e\in E(X)}(|G^{e}|-1),$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^{x} stands for the stabiliser of $x\in V(X)\cup E(X)$ in G and $|G^{x}|$ is the order of a stabiliser.

Groups acting on a graph with edge reversing

Theorem 3 (M., 2013)

Let X be a graph of genus g and G is a finite group acting on X, possibly with edge reversing. Denote by $\gamma = g(X/G)_{tail}$ genus of the factor graph $(X/G)_{tail}$. Then

$$g-1=|G|(\gamma-1)+\sum_{v\in V(X)}(|G^{v}|-1)-\sum_{e\in E(X)}(|G^{e}|-1)+\sum_{e\in E^{inv}(X)}|G^{e}|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^{x} is the stabiliser of $x \in V(X)\cup E(X)$ in G, and $E^{inv}(X)$ is the set of invertibile edges of X.

Harmonic group action on graphs

Let X be a finite connected multigraph without loops.

Definition

A group $G < \mathrm{Aut}(X)$ acts *harmonically* on a graph X if and only if it acts freely on the set of arcs of X.

We have the following observation.

Observation

If group G acts harmonically on a graph X then the canonical projection $X \to X/G$ is a branched covering.

Harmonic group action on graphs

Let finite group G acts *harmonically* on a graph X. Then $|G^e| = 1$ for each $e \in E(X)$. We have the following corollary from Theorem 3 (compare with Baker-Norine, 2009 and Corry, 2011).

Corollary

Let X be a graph of genus g and G is a finite group acting on X harmonically, possibly with edge reversing. Denote by $g(X/G)_{free}$ genus of the factor graph $(X/G)_{free}$. Then

$$g - 1 = |G|(g(X/G)_{free} - 1) + \sum_{v \in V(X)} (|G^v| - 1) + |E^{inv}(X)|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^v is the stabiliser of $v \in V(X)$ in G, and $E^{inv}(X)$ is the set of invertible edges of X.

Harmonic group action on graphs

Recall some classical results for Riemann surface theory. For each $g \geq 2$ define

$$N(g) := \max\{|\mathrm{Aut}(S_g)| : \ S_g \text{ is a compact Riemann surface of genus } g\}.$$

Then

$$8(g+1) \leq N(g) \leq 84(g-1),$$

and these bounds are sharp in the sense that both the upper and lower bound are attained for infinitely many values of g. The upper bound was found by Hurwitz (1893). The lower bound was independently obtained by R. Accola (1968) and C. Maclachlan (1969).

Denote by $N(g)$ maximum size of a finite group acting harmonically on a graph of genus $g \geq 2$.

Theorem (Scott Corry, 2011)

For $g \geq 2$ we have

$$4(g-1) \leq N(g) \leq 6(g-1).$$

The upper and lower bound are attained for infinitely many values of g.

Recent paper by Scott Corry (2013) states that maximal graph groups G with $|G| = 6(g-1)$ are exactly the finite quotients of the modular group $\Gamma =< x, y | x^2 = y^3 = 1 >$ of size at least 6.

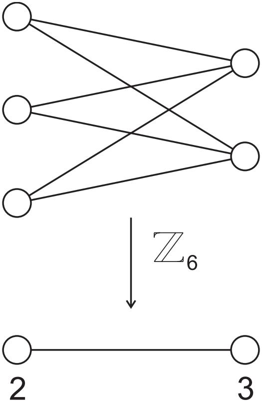
Wiman's theorem

Klein's quartic curve, $x^3y + y^3z + z^3x = 0$, admits the group $\mathrm{PSL}_2(7)$ as its full group of conformal automorphisms. It is a genus three curve which has exactly $168 = 84(3-1)$ automorphisms. This is the curve of smallest genus realising the upper bound $84(g-1)$ on the order of a group of conformal automorphisms of a curve of genus $g > 1$, given by A. Hurwitz in 1893. Around the same time, A. Wiman (1895) characterised the curves $w^2 = z^{2g+1} - 1$ and $w^2 = z(z^{2g} - 1), g > 1$, as the unique curves of genus g admitting cyclic automorphism groups of the largest and the second largest possible order ($4g+2$ and $4g$, respectively). The modern proof of these and similar results is contained in the paper by K. Nakagawa (1984).

We obtain the following discrete version of the Wiman theorem.

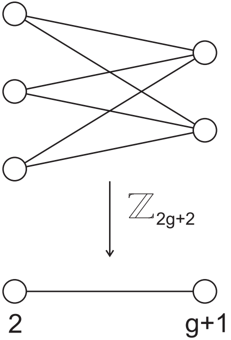
Theorem 4 (A. Mednykh and I. Mednykh, 2013)

Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on X. Then $N \leq 2g+2$. The upper bound $N = 2g+2$ is attained for any even g. In this case, the signature of orbifold $X/\mathbb{Z}_N$ is $(0; 2, g+1)$, that is, $X/\mathbb{Z}_N$ is a tree with two branch points of order 2 and $g+1$, respectively.



$K_{2,3}$ - genus 2 graph

Wiman's theorem



$K_{2,g+1}$ - genus g graph

g - even

Wiman's theorem

The second and the third largest cyclic groups are given by

Theorem 5 (A. Mednykh and I. Mednykh, 2013)

Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on X. Let $N < 2g + 2$ then $N \leq 2g$. The upper bound $N = 2g$ is attained only in the following cases:

(i) $N = 2g$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 2, 2g), g \geq 2$;

(ii) $N = 12$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 4)$, $g = 6$.

Moreover, the upper bound $N = 2g - 1$ is attained only in two cases:

(iii) $N = 3$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 3)$, $g = 2$;

(iv) $N = 15$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 5)$, $g = 8$.

Hyperelliptic graphs

A graph X is said to be hyperelliptic, if it has a two-fold harmonic map on a tree. The corresponding covering transformation is called *hyperelliptic involution*. It is unique if the genus of graph X is grater then one. We note that any graph of genus two without loops and bridges is hyperellptic (Baker-Norine, 2007).

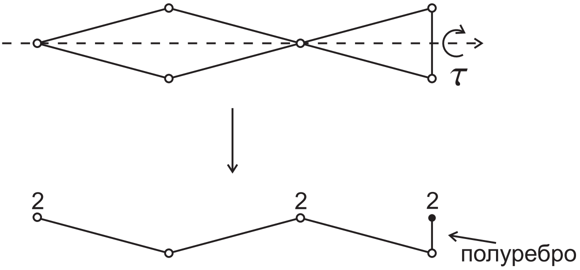


Fig. 2: Hyperelliptic graph of genus 2

Hyperelliptic graphs

A graph is said to be $\gamma-$hyperelliptic if it is a two fold branched covering of a genus γ graph. The corresponding covering involution is called $\gamma-$hyperelliptic.
The aim of the next theorem is to provide two criteria for involution τ acting on a graph X of genus g to be $\gamma-$hyperelliptic.

Hyperelliptic graphs

Theorem

Let X be a finite connected graph of genus g. Consider an involution τ acting free on the set of directed edges of X. Denote by τ_* the induced action of τ on the first homology group $H_1(X)$. Suppose that τ has a least one fixed vertex on X. Then the following conditions are equivalent:

(i) *The genus of factor graph $X/\langle\tau\rangle$ is equal to γ.*

(ii) *There is a basis $a_1, a_2, \ldots, a_g$ in the homology group $H_1(X)$ whose $g-2\gamma$ elements are invertible and the others 2γ split into γ interchangeable pairs under the action of τ_*.*

(iii) $tr_{H_1(X)}(\tau_*) = 2\gamma - g$.

If τ acts fixed point free on X then genera g and γ are related by the Schreier formula $g - 1 = 2(\gamma - 1)$.

Oikawa theorem

In 1956 Kotaro Oikawa proved the following theorem.

Theorem (Oikawa, 1956)

Let S_g be a closed Riemann surface of genus g and A is a finite subset of S_g consisting of $|A| \geq 1$ elements. Suppose that $2g - 2 + |A| > 0$ and G is a group of conformal automorphisms of S_g leaving the set A invariant. Then

$$|G| \leq 12(g-1) + 6|A|.$$

In the next section we find a discrete version of the Oikawa's. Again, the key point of the proof is the Riemann-Hurwitz relation.

Our result for graphs is the following theorem.

Theorem 6 (R. Nedela, A. Mednykh, 2013)

Let X be a graph of genus g and A is a subset of vertices of X consisting of $|A| \geq 1$ elements. Suppose that $g - 1 + |A| > 0$ and G is a finite group acting on X harmonically and leaving the set A invariant. Then

$$|G| \leq 2(g-1) + 2|A|.$$

The upper bound is sharp and is attained for arbitrary large values of g and $|A|$. So, at least infinitely many often.

Two Arakawa's theorems

Now our aim is to find discrete versions of two Arakawa's theorems (2000).

The first one states that if G be a finite group of automorphisms of a compact Riemann surface X of genus $g \geq 2$ and A and B are two disjoint G-invariant subsets of X of the orders $|A| \geq |B| \geq 1$ then $|G| \leq 8(g-1) + |A| + 4|B|$.

The second theorem asserts that if A, B and C are three disjoint the G-invariant subsets of X with $|A| \geq |B| \geq |C| \geq 1$ then $|G| \leq 2(g-1) + |A| + |B| + |C|$.

We present a discrete version of the first Arakawa's theorem by the following theorem.

Theorem 7 (R. Nedela, A. Mednykh and I. Mednykh 2013)

Let X be a graph of genus $g \geq 2$ and A and B are two disjoint subsets of vertices of X of the orders $|A| \geq |B| \geq 1$. Suppose that G is a finite group acting harmonically on X and leaving the sets A and B invariant. Then

$$|G| \leq \frac{3(g-1) + |A| + 3|B|}{2}.$$

Again, the upper bound is sharp and is attained for arbitrary large values of $g, |A|, |B|$ and $|C|$.

Two Arakawa's theorems

A discrete version of the second Arakawa's theorem is given by the following theorem.

Theorem 8 (R. Nedela, A. Mednykh and I. Mednykh, 2013)

Let X be a graph of genus $g \geq 2$ and A, B and C are three disjoint subsets of vertices of X of the orders $|A| \geq |B| \geq |C| \geq 1$. Suppose that G is a finite group acting harmonically on X and leaving the sets A, B and C invariant. Then

$$|G| \leq \frac{g - 1 + |A| + |B| + |C|}{2}.$$

As in the two previous theorems, the upper bound is sharp and is attained for arbitrary large values of $g, |A|, |B|$ and $|C|$.